

Discrete-time detection modeling for unsaturated ocean acoustic propagation

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The basic problem in ocean acoustic detection is formulated under the assumption of unsaturated sound propagation. The latter essentially amounts to a constant signal plus Gaussian noise. Detection is defined as occurring whenever ρ , the root mean square pressure at the receiver, exceeds a specified threshold level ρ_0 . A two-state, discrete-time Markov model is derived, and closed-form expressions for the probability mass functions of the number of time steps separating two successive detections (interarrival time) or one detection and the first subsequent "downcrossing" (holding time) are presented. Expressions for the joint probability density function of ρ at two different points in time are obtained and used to determine the relevant one-step transition probabilities of the Markov model. Sample results using the model are finally presented.

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INTRODUCTION

In general, the quadrature components of the envelope of a narrow-band ocean acoustic multipath process are given by¹

$$X = \sum_{n=1}^N (r_n \cos \theta_n + N_x^{(n)}), \quad (1)$$

$$Y = \sum_{n=1}^N (r_n \sin \theta_n + N_y^{(n)}),$$

where

N = number of independent paths between source and receiver,

r_n = the amplitude of the n th path,

θ_n = the phase of the n th path

$N_x^{(n)}, N_y^{(n)}$ = zero-mean, uncorrelated Gaussian additive noise for the n th path.

Furthermore, the envelope and the phase of the total signal are defined as:

$$\rho = (X^2 + Y^2)^{1/2}, \quad (2)$$
$$\phi = \tan^{-1}(Y/X).$$

At short ranges and low frequencies, or for stable channels, the propagation is said to be *unsaturated* and the probability density function (PDF) of ρ is Rician² and independent of the number of paths.¹ (In Sec. I the distributions of ρ and its phase ϕ are presented.)

At sufficiently long ranges and/or high frequencies, the propagation is *fully saturated*, which means that ϕ , the phase of ρ , can be characterized as a random variable uniformly distributed between 0 and 2π , or each path has a phase θ_n that is normally distributed with a standard deviation $> 2\pi$. In this regime when $N > 4$ and the single path amplitudes r_n are approximately equal, phase random multipath propaga-

tion is obtained. It has been found³⁻⁵ that the envelope ρ of a fully saturated phase random process with additive Gaussian noise obeys a Rayleigh PDF. Moreover, several other statistics and joint PDF's for the phase random process have been obtained, and are presented in a comprehensive summary by Mikhalevsky.⁶

In intermediate ranges, where the signal experiences enough perturbations in the channel so that each θ_n can be characterized as a Gaussian random variable but with a standard deviation $< 2\pi$, *partially saturated* propagation is obtained. The frequency/range boundaries between the unsaturated, partially saturated, and fully saturated regimes are dependent upon the ocean dynamics or boundary dynamics of the propagation channel, as well as the magnitude of any relative source-receiver motion. In Ref. 1, the envelope statistics for signals in the partially saturated regime were presented. As the variance of the single path phase goes to zero, or becomes large, the PDF's converge to the unsaturated and fully saturated results, respectively.¹²

In previous publications^{7,8} of the authors, continuous and discrete-time detection models using the results of phase random acoustic propagation³⁻⁶ have been formulated. "Detection" was defined as an upcrossing of random variable ρ (the root mean square pressure at the passive sonar receiver) over a specified threshold ρ_0 . A continuous-time model was first developed for obtaining the PDF's of the time between a detection and the first subsequent downcrossing through ρ_0 (holding time). The model was then compared with the extensively used (λ, σ) model and with available acoustic data. This model was seen to exhibit similar long-term behavior but markedly different short term characteristics as compared with the (λ, σ) model, a fact which is due to the memory of the process. Comparison with data has demonstrated, in most cases, a significantly improved prediction capability over the (λ, σ) model.

Subsequently, a two-state model and a four-state discrete-time Markov detection model were developed, and closed-form expressions for the probability mass functions of the corresponding interarrival and holding times were derived. The results obtained using the latter models were favorably compared with both the continuous-time models and the data, the greatest improvement over the continuous-time models lying in the much lower computational effort involved.

The purpose of this letter is to develop acoustic detection models for the *unsaturated* case. Such models are derived in the following section, first for the memoryless case and then for the general "memory" Markov case. A model has also been proposed⁹ for the partially saturated case, but results are yet to be confirmed in other than limiting cases.

I. ANALYTICAL FORMULATION

The probability density functions for the root mean square pressure ρ and its phase ϕ for the unsaturated process are derived in Ref. 1. The density of ρ is Rician:

$$f_\rho(\rho) = \frac{\rho}{\sigma_N^2} \exp\left(-\frac{\rho^2 + R_s^2}{2\sigma_N^2}\right) I_0\left(\frac{\rho R_s}{\sigma_N^2}\right), \quad 0 < \rho < \infty, \quad (3)$$

where

R_s = the magnitude of the constant signal vector,

I_0 = modified Bessel function of the first kind of zero order,

$\sigma_N^2 = N\sigma_{N_x}^2 = N\sigma_{N_y}^2$, where

$\sigma_{N_x}^2, \sigma_{N_y}^2$ = the variances of $N_x^{(n)}$ and $N_y^{(n)}$, respectively, as defined in the introduction.

The density of ϕ is

$$f_\phi(\phi) = \frac{1}{2\pi\sigma_N^2} \exp\left(-\frac{R_s^2}{2\sigma_N^2}\right) \left[\sigma_N^2 + \sigma_N M_s \left(\frac{\pi}{2}\right)^{1/2} \times \exp\left(\frac{M_s^2}{2\sigma_N^2}\right) \left[1 - \Phi\left(\frac{M_s}{(2\sigma_N^2)^{1/2}}\right) \right] \right], \quad (4)$$

where

$$M_s = \mu_x \cos \phi + \mu_y \sin \phi, \quad (5)$$

μ_x, μ_y = the quadrature components of R_s ($R_s^2 = \mu_x^2 + \mu_y^2$), and

$$\Phi(x) = \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

In our previous Markov modeling of the phase random process (fully saturated sound propagation), a two-state model and a four-state model were developed.⁷⁻⁸ Comparison with data has revealed that both models, when properly calibrated, yield very satisfactory results, the two-state being consistently as accurate as the four-state model.⁸

We will henceforth restrict ourselves in developing a two-state Markov model for the unsaturated process of the general form shown in Fig. 1,

where "U" = "up" state, defined by $\rho \geq \rho_0$,

"D" = "down" state, defined by $\rho < \rho_0$,

$a = \text{prob}(\rho_2 < \rho_0 | \rho_1 \geq \rho_0)$,

$b = \text{prob}(\rho_1 \geq \rho_0 | \rho_2 < \rho_0)$.

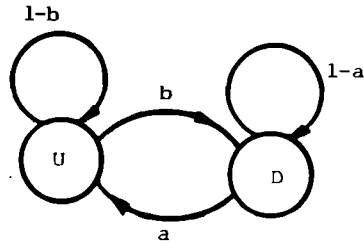


FIG. 1. Two-state discrete-time Markov model.

In the *memoryless* case, $a + b = 1$, and

$$\begin{aligned} a &= \text{prob}(\rho_2 < \rho_0 | \rho_1 \geq \rho_0) = \text{prob}(\rho_2 < \rho_0 | \rho_1 < \rho_0) \\ &= \text{prob}(\rho_2 < \rho_0) \end{aligned}$$

or

$$a = \int_0^{\rho_0} f_\rho(\rho) d\rho = 1 - Q_1\left(\frac{R_s}{\sigma_N}, \frac{\rho_0}{\sigma_N}\right) \quad (6)$$

with $f_\rho(\rho)$ as in (3), and the generalized Q -function is defined¹⁰ as

$$Q_M(\alpha, \beta) = \int_\beta^\infty \xi \left(\frac{\xi}{\alpha}\right)^{M-1} \exp\left(-\frac{\alpha^2 + \xi^2}{2}\right) I_{M-1}(\alpha\xi) d\xi, \quad M = 1, 2, \dots \quad (7)$$

It can be shown that

$$\begin{aligned} 1 - Q_1(\alpha, \beta) &= \exp\left(-\frac{\alpha^2 + \beta^2}{2}\right) \\ &\times \sum_{k=0}^{\infty} \frac{(\alpha^2/2)^k}{k!} \sum_{n=k+1}^{\infty} \frac{(\beta^2/2)^n}{n!}. \end{aligned} \quad (8)$$

Rappaport¹⁰ presents several approximations for Q and its complement, which is less cumbersome to evaluate exactly in the computer than using (8). Note: Rice² presents the equivalent result:

$$a = \int_0^{\rho_0} f_\rho(\rho) d\rho = \exp\left(-\frac{\rho_0^2 + R_s^2}{2\sigma_N^2}\right) \sum_{n=1}^{\infty} \left(\frac{\rho_0}{R_s}\right)^n I_n\left(\frac{\rho_0 R_s}{\sigma_N^2}\right), \quad (9)$$

where

$$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k} \frac{1}{k!(n+k)!}.$$

From (9), we can proceed to evaluate the probability mass functions for the interarrival and holding time. In general, these PMF's take the form⁸

$$P_H(k) = (1-a)^{k-1} a, \quad k = 1, 2, \dots (\text{holding time}), \quad (10)$$

$$\begin{aligned} P_I(n) &= [ab/(a-b)] [(1-b)^{n-1} - (1-a)^{n-1}], \\ n &= 2, 3, \dots (\text{interarrival time}). \end{aligned} \quad (11)$$

In the memoryless case, (11) becomes

$$P_I(n) = \frac{a(1-a)}{(1-2a)} [(1-a)^{n-1} - a^{n-1}], \quad n = 2, 3, \dots \quad (12)$$

In the nontrivial case ($a + b \neq 1$) of the Markov model of Fig. 1, the calculation of the transition probabilities requires knowledge of the joint density function $f_{\rho, \phi}(\rho_1, \rho_2)$,

where $\rho_1 = \rho(t)$ and $\rho_2 = \rho(t + T)$. This second-order density has already been derived in a rather general form by Middleton's¹¹ treatment of the statistical properties of additive narrowband signal and normal noise processes.

Using Middleton's results, and after extensive algebraic manipulations, we obtain

$$f_{\rho_1, \rho_2}(\rho_1, \rho_2) = \frac{\rho_1 \rho_2}{\sigma_N^2(1 - \rho_0^2)} \exp\left(-\frac{\rho_1^2 + \rho_2^2}{2\sigma_N^2(1 - \rho_0^2)}\right) \times \exp\left(\frac{A_0^2}{\sigma_N^2(1 + \rho_0)}\right) \cdot \sum_{m=0}^{\infty} \epsilon_m I_m\left(\frac{r_0 \rho_1 \rho_2}{\sigma_N^2(1 - \rho_0^2)}\right) \times I_m\left(-\frac{A_0 \rho_1}{\sigma_N^2(1 + \rho_0)}\right) I_m\left(\frac{A_0 \rho_2}{\sigma_N^2(1 + \rho_0)}\right). \quad (13)$$

It is reasonable to expect that (13) will reduce, for $t \rightarrow \infty$, to the product of $f_{\rho_1}(\rho_1) f_{\rho_2}(\rho_2)$, i.e.,

$$f_{\rho_1, \rho_2}(\rho_1, \rho_2) \xrightarrow{T \rightarrow \infty} \frac{\rho_1 \rho_2}{\sigma_N^4} \exp\left(-\frac{\rho_1^2 + \rho_2^2 + 2A_0^2}{2\sigma_N^2}\right) \times I_0\left(\frac{A_0 \rho_1}{\sigma_N^2}\right) I_0\left(\frac{A_0 \rho_2}{\sigma_N^2}\right). \quad (14)$$

Assuming $k_0 \rightarrow 0$ for $T \rightarrow \infty$ (uncorrelatedness), (13) gives

$$f_{\rho_1, \rho_2}(\rho_1, \rho_2) \xrightarrow{T \rightarrow \infty} \frac{\rho_1 \rho_2}{\sigma_N^4} \exp\left(-\frac{\rho_1^2 + \rho_2^2 + 2A_0^2}{2\sigma_N^2}\right) \times I_0\left(\frac{A_0 \rho_1}{\sigma_N^2}\right) I_0\left(\frac{A_0 \rho_2}{\sigma_N^2}\right). \quad (15)$$

$$\text{Since } I_m(x) = \frac{(x/2)^m}{m!} \rightarrow \begin{cases} 0, & m \neq 0 \\ 1, & m = 0, \end{cases} \quad (16)$$

Eq. (15) can be rewritten as

$$f_{\rho_1, \rho_2}(\rho_1, \rho_2) \xrightarrow{T \rightarrow \infty} \frac{\rho_1}{\sigma_N^2} \exp\left(-\frac{\rho_1^2 + A_0^2}{2\sigma_N^2}\right) I_0\left(\frac{A_0 \rho_1}{\sigma_N^2}\right) \frac{\rho_2}{\sigma_N^2} \times \exp\left(-\frac{\rho_2^2 + A_0^2}{2\sigma_N^2}\right) I_0\left(\frac{A_0 \rho_2}{\sigma_N^2}\right),$$

$$\text{i.e., } f_{\rho_1, \rho_2}(\rho_1, \rho_2) \xrightarrow{T \rightarrow \infty} f_{\rho_1}(\rho_1) f_{\rho_2}(\rho_2). \quad (17)$$

We can now proceed to evaluate the one-step transition probabilities of the Markov model.

$$P_{12} = a = \int_0^{\rho_0} \int_0^{\rho_0} f_{\rho_1, \rho_2}(\rho_1, \rho_2) d\rho_1 d\rho_2 \int_{\rho_0}^{\infty} f_{\rho}(\rho) d\rho \quad (18)$$

$$P_{21} = b = \int_0^{\rho_0} \int_0^{\rho_0} f_{\rho_1, \rho_2}(\rho_1, \rho_2) d\rho_1 d\rho_2 \int_0^{\rho_0} f_{\rho}(\rho) d\rho \quad (19)$$

(and $P_{11} = 1 - P_{12}$, $P_{22} = 1 - P_{21}$).

The double integrals in (18) and (19) can be evaluated as functions of

$$\Sigma_1 = \int_0^{\rho_0} \int_0^{\rho_0} f_{\rho_1, \rho_2}(\rho_1, \rho_2) d\rho_1 d\rho_2. \quad (20)$$

Although Σ_1 is symmetric with respect to ρ_1 and ρ_2 , it cannot

be expressed as a product of one function of ρ_1 and one of ρ_2 . Instead, we can rewrite (20), taking (13) into account, as follows:

$$\Sigma_1 = \int_0^{\rho_0} \frac{\rho_1}{\sigma_N^4(1 - \rho_0^2)} \exp\left(-\frac{\rho_1^2}{2\sigma_N^2(1 - \rho_0^2)} - \frac{A_0^2}{\sigma_N^2(1 + \rho_0)}\right) \int_0^{\rho_0} \rho_2 \exp\left(-\frac{\rho_2^2}{2\sigma_N^2(1 - \rho_0^2)}\right) \times \sum_{m=0}^{\infty} I_m\left(\frac{\rho_0 \rho_1 \rho_2}{\sigma_N^2(1 - \rho_0^2)}\right) \times I_m\left(\frac{A_0 \rho_1}{\sigma_N^2(1 + \rho_0)}\right) I_m\left(\frac{A_0 \rho_2}{\sigma_N^2(1 + \rho_0)}\right) d\rho_2 d\rho_1. \quad (21)$$

In the above, A_0 is identical to R_S of Eq. (3).

Having evaluated Σ_1 , Eqs. (18)–(19) can be expressed as

$$P_{12} = (P_u - \Sigma_1)/(1 - P_u), \quad (22)$$

$$P_{21} = 1 - \Sigma_1/P_u, \quad (23)$$

$$P_{11} = 1 - P_{12}, \quad P_{22} = 1 - P_{21}, \quad (24)$$

where P_u is the (unconditional probability of ρ being less than ρ_0). Efforts to simplify the evaluation of Σ_1 in Eq. (21) were not successful. The double numerical integration of a function involving the infinite sum of products of three modified Bessel functions was expected to and did actually produce computational problems (excessive CPU time). These were partially alleviated using the asymptotic properties of the Bessel functions involved in determining the tolerances employed in terminating the evaluation of the infinite summations. Still, for extreme (that is, too small or too large) detection thresholds, the computational effort is unacceptably large. However, this is not expected to be a problem in practice since we do not need to use such extreme thresholds—in fact, they result in memoryless Markov models, and the problem does not exist, since the evaluation of the relevant one-step transition probabilities just requires the knowledge of the unconditional distributions.

II. IMPLEMENTING THE MODELS

The results obtained using our model with a variety of—fictitious but hopefully appropriate—inputs for the parameters involved, namely σ_N^2 , ν , R_S , the time step, and the detection threshold.¹²

Figures 2–3 present typical results using $\sigma_N^2 = 1.58$, $R_S = 2.23$, ($R_S^2 = 5$) $\nu = 0.2$ Hz, a time step of 0.4 s, and thresholds of $\rho_0 = 1.58$. σ_N^2 and R_S were picked from an unsaturated example,¹ and the rest were chosen by the authors and are more or less arbitrary. It is seen¹² that, although the density of the holding time is very sensitive to the magnitude of the detection threshold, the density of the interarrival time is much less so, being almost independent of the threshold. This reminds us of what we would get in a pure sinusoidal signal situation, where we have a constant interarrival time (equal to the period of the sinusoidal signal) but different holding times for each threshold. Such a result was not observed in our previous study of the detection process

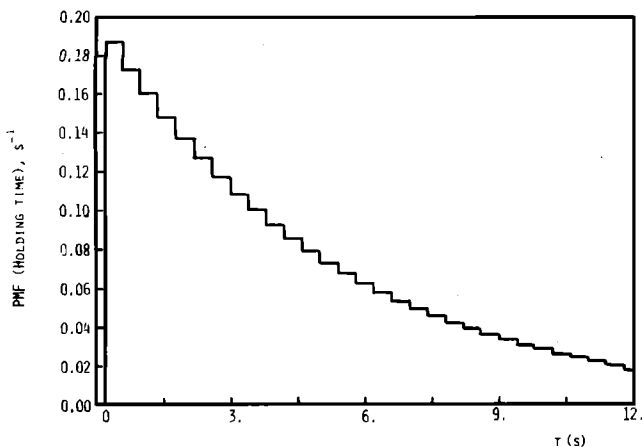


FIG. 2. Holding time, $\rho_0 = 1.58$, $T = 0.4$ s, $R_S = 2.23$.

using the *phase random* model for ocean propagation.⁸ In the unsaturated case, ρ is obviously not a strict sinusoid and, hence, we do not get the above δ -function densities for the interarrival and holding times. However, we also show¹² that the timing of detection events (i.e., the density of the interarrival times) is almost independent of the detection threshold. This threshold makes its presence felt only in the densities of the holding time, in which we quite obviously have shorter holding times for higher thresholds. We have also obtained results¹² demonstrating the relative insensitivity of the above results to changes in the time step T .

III. CONCLUSIONS

In this letter, an analytical model for the unsaturated acoustic detection process was presented, and probability mass functions for the interarrival and holding times were

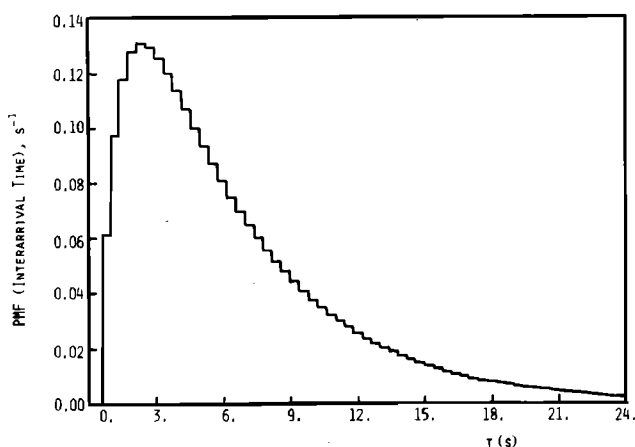


FIG. 3. Interarrival time, $\rho_0 = 1.58$, $T = 0.4$ s, $R_S = 2.23$.

derived. The unsaturated mode of acoustic propagation was seen to exhibit different characteristics than the previously developed phase random acoustic detection models. A major difference between these two modes of acoustic propagation lies in the narrower (for the unsaturated case) distribution of ρ , which approaches a normal density as R_S grows large. A more striking difference lies in the relative independence of the interarrival time PMF to the detection threshold ρ_0 for the unsaturated case, a property not observed in the fully saturated models.

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